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THERMOELASTIC DAMPING

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L. E. Goodman, C. S. Chang, A. R. Robinson, authors)

NOTICES

FOREWORD

This report was prepared by the Department of Aeronautics and Engineering Mechanics of the University of Minnesota under USAF Contract No. AF 33(657)-7453. This contract was initiated under Project No. 7351, "Metallic Materials," Task No. 735106, "Behavior of Metals." The work was administered under the direction of the Metals and Ceramics Laboratory, Directorate of Materials and Processes, Deputy for Technology, Aeronautical Systems Division, with Mr. D. M. Forney, Jr. acting as project engineer.

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ABSTRACT

The non-linear coupled field equations of thermoelasticity are herein solved for the quasi-static behavior of a solid bounded by two parallel planes. The mechanical energy converted to heat during each cycle of the loading process is explicitly evaluated and its spectral variation computed. From this exact solution an approximate, but quite general, expression for the thermoelastic energy dissipated in solids as a result of elastic deformation is developed.

This technical documentary report has been reviewed and is approved.

W. J. Trapp

W. J. TRAPP
Chief, Strength and Dynamics Branch
Metals and Ceramics Laboratory
Directorate of Materials and Processes

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I. INTRODUCTION

Thermoelastic damping as one of the causes of energy dissipation in engineering materials has been studied by a number of authors. Among them Zener (1,2),* Lessen (3,4), and Biot (5) may be mentioned. In his treatment of thermoelasticity as a part of general theory of irreversible thermodynamics, Biot works out an example of thermoelastic damping in an ideal material.** The equation of heat conduction in his case is developed by a slightly different approach from that used by Jeffreys (6). Biot considers the unit volume, which is fixed in space, as the fundamental thermodynamic system. Unlike Jeffreys, however, Biot does not find it necessary to use the conservation of mass requirement. A linearization of the entropy expression leads to a linear heat conduction equation in Biot's treatment. Quantities of transport by the transfer of mass across the system boundary are not considered in (5) and do not arise in (6).

In this report, intensive quantities per unit mass are used. The boundary of the fundamental system, the unit mass, is varying, but it always contains the same mass. The usual conservation of mass relationship is used and the equations of motion are shown to be consistent with the unit mass system. The second law of thermodynamics is postulated in deriving the rate of entropy variation in an elastic solid. It is then shown that energy dissipation is due explicitly to the irreversibility of the heat conduction process. The non-linearity in the heat equation is then shown to be necessary in order that predictions of the theory be consistent with the energy conservation requirement. This is a point of interest in that a linear thermodynamic consideration leads to a non-linear field equation. It is also a point of some importance in the calculation of energy dissipation per cycle in the steady-state vibration of a system because the amount of heat ejected by the system is entirely lost by the linearization.

An example which is simple enough to be worked out in mathematical detail is solved and the energy dissipation calculated. An approximate theory is then introduced. The result is shown to be very accurate, as compared to that obtained by the exact method.

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* Numbers in parentheses refer to the references on page 30.

** Sometimes the treatment of thermoelastic damping in idealized materials is erroneously considered as to explain actual damping in real, imperfect materials.

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II. THERMODYNAMICS OF ELASTIC SOLIDS

In the consideration of the coupled phenomenon of mechanical deformations and the conduction of heat in an idealized elastic material, we shall choose as thermodynamic state variables the six components of the stress tensor, σ_{ij} , the six components of the strain tensor, ϵ_{ij} , and the local absolute temperature, T. We are dealing with a perfect continuum for which the above state variables are, in general, functions of the space coordinates, x_i , as well as the time, t.

An isotropic solid is defined as elastic in this paper if the equations of state,

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk} - (3\lambda + 2\mu)\alpha\delta_{ij}\theta \quad , \qquad (2\text{-}1)$$

$$i, j, k = 1, 2, \text{ or } 3 ,$$

are observed everywhere within the solid. In equation (2-1), θ denotes the local temperature in excess of the uniform reference temperature T_o which is chosen together with the reference unstressed and undeformed state, i.e., $\sigma_{ij} = \epsilon_{ij} = 0$ at T_o; λ and μ denote the Lamé constants of elasticity under isothermal conditions; α denotes the linear coefficient of thermal expansion and

$$\delta_{ij} = 1 \qquad \text{if} \quad i = j \;,$$
$$= 0 \qquad \text{if} \quad i \neq j \;.$$

The basic relationships to be studied are the equations which state the conservation of momentum, mass, and energy, and the second law of thermodynamics for the irreversible process written in the form of an equality. Intensive quantities are to be used in these equations. These quantities are, strictly speaking, material point functions, but the unit mass may be used to help visualize a physical thermodynamic system, e.g., the value of the specific entropy at a point may be regarded as the total entropy of a unit mass centered on the point in question. Since the boundary of this fundamental system is deforming with it, there is no mass transfer across the boundary and the question of quantities of transport need not arise.

The conservation of momentum is stated by the well-known equations of motion

$$\sigma_{ji,j} + \rho F_i = \rho \ddot{u}_i . \tag{2-2}$$

In the last equation ρ is the mass density of the solid; F_i, the bodily force per unit mass; and u_i, the displacement vector. We have also adapted the convention of denoting differentiation with respect to space coordinates by a comma, and differentiation with respect to time by a dot over the quantity. It is perhaps worthwhile to remark that equations (2-2) are satisfied by the medium in its deformed state, and hence, the use of Eulerian coordinates are indicated. This is consistent with the use of a deforming unit mass system. In addition to the equations of motion, appropriate stress, or displacement, boundary conditions must be satisfied.

The conservation of mass can be stated in the simple form

$$\rho = \frac{\rho_o}{1+\epsilon} \tag{2-3}$$

where ϵ is the cubical dilatation and ρ_o is the density of the material at zero strain. In addition, there are the usual compatibility requirements on the strains such that no material discontinuity may occur.

The conservation of energy may be stated by the equation

$$\frac{\partial U}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial t}(\dot{u}_i \dot{u}_i) + F_i \dot{u}_i + \frac{1}{\rho}(\sigma_{ij} u_i)_{,j} - \frac{1}{\rho} J_{i,i} \tag{2-4}$$

The terms in the above equation represent, respectively, from left to right, the rate of change of internal energy, the rate of change of specific kinetic energy, the rate of work being done by the bodily forces, the rate of work being done by the surface tractions, and the divergence of a heat flux, expressed for the volume occupied by a unit mass at a particular time. The expression for the work done by the surface tractions can be derived by integrating the scalar product of the surface traction and the velocity of a surface element over the boundary of a unit mass, and by applying the divergence theorem after converting the surface traction into stresses

$$\int_{A(\text{unit mass})} \overset{\nu}{T}_i \, \dot{u}_i \, dA = \int_A \sigma_{ij} \, \dot{u}_i \, \nu_j \, dA$$

$$= \int_V (\sigma_{ij} \, \dot{u}_i)_{,j} \, dv$$

We can eliminate the kinetic energy term from equation (2-4) by scalarly multiplying the equations of motion (2-2) by the vector $\dot{u}_i$ and substituting the results into equation (2-4), obtaining

$$\frac{\partial U}{\partial t} = \frac{\sigma_{ij}}{\rho} \frac{\partial \epsilon_{ij}}{\partial t} - \frac{1}{\rho} J_{i,i} \tag{2-5}$$

The strain-defining relationships

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2-6}$$

have also been used. The substitutions

$$\epsilon_{11} = \epsilon_1, \epsilon_{22} = \epsilon_2, \epsilon_{33} = \epsilon_3, 2\epsilon_{23} = \epsilon_4, 2\epsilon_{31} = \epsilon_5, 2\epsilon_{12} = \epsilon_6 \quad ,$$

$$\frac{\sigma_{11}}{\rho} = \sigma_1, \frac{\sigma_{22}}{\rho} = \sigma_2, \frac{\sigma_{33}}{\rho} = \sigma_3, \frac{\sigma_{23}}{\rho} = \sigma_4, \frac{\sigma_{31}}{\rho} = \sigma_5, \frac{\sigma_{12}}{\rho} = \sigma_6 \quad .$$

may be used to transform equation (2-5) into (using differentials) the equation

$$du = \sigma_i \, d\epsilon_i + dq \tag{2-7}$$

with dq indicating a differential heat added to the unit mass, i.e.,

$$\rho \frac{\partial q}{\partial t} = - J_{i,i} \quad . \tag{2-8}$$

The equations of state in terms of the new variables ϵ_i and σ_i become

$$\rho\,\sigma_i = 2\mu\,\epsilon_i + \lambda\,\epsilon - (3\lambda + 2\mu)\,a\,\theta\ , \qquad \text{for} \quad i = 1, 2 \text{ or } 3\ ; \tag{2-9}$$

and

$$\rho\,\sigma_i = \mu\,\epsilon_i \qquad \text{for} \quad i = 4, 5 \text{ or } 6\ . \tag{2-10}$$

In order to derive the expression for the change of specific entropy, denoted by the symbol s, the following second law of thermodynamics is postulated:

$$T\,ds = dq\ , \tag{2-11}$$

the relationship for a reversible process. This postulate relies on the argument that the only type of "driving force" besides the temperature gradient are the mechanical forces; and that an elastic body eventually comes to rest under a sustained load, unlike the continuous flow of heat under its "driving force," the temperature gradient. The behavior of an elastic body is also unlike the viscous flow of a fluid under a sustained shearing force.

The acceptance of this relationship as applicable outside of thermodynamic equilibrium is, of course, an essential part of the usual theory of elastic solids. Also, in accepting equation (2-11), we assume that the specific entropy depends explicitly on the state variables only.

We are now ready to derive an expression for the specific entropy. Writing

$$dq = \tau_i\,d\epsilon_i + c\,dT \tag{2-12}$$

where the τ_i and c are as yet undetermined functions of the state variables, and substituting this expression into equations (2-7) and (2-11), we obtain

$$dU = (\sigma_i + \tau_i)\,d\epsilon_i + c\,dT\ , \tag{2-13}$$

and

$$ds = \frac{\tau_i}{T}\, d\epsilon_i + \frac{c}{T}\, dT \ . \tag{2-14}$$

Since U, as well as s, is a function of state only, dU, as well as ds, is a perfect differential. This will furnish sufficient relationships to determine the nature of τ_i and c. We have from equation (2-13),

$$\frac{\partial(\sigma_i + \tau_i)}{\partial\epsilon_j} = \frac{\partial(\sigma_j + \tau_j)}{\partial\epsilon_i} \quad , \tag{2-15}$$

and

$$\frac{\partial(\sigma_i + \tau_i)}{\partial T} = \frac{\partial c}{\partial\epsilon_i} \quad , \tag{2-16}$$

and from equation (2-14),

$$\frac{\partial\tau_i}{\partial\epsilon_j} = \frac{\partial\tau_j}{\partial\epsilon_i} \quad , \tag{2-17}$$

and

$$\frac{\partial\left(\frac{\tau_i}{T}\right)}{\partial T} = \frac{1}{T}\,\frac{\partial c}{\partial\epsilon_i} \quad . \tag{2-18}$$

From equations (2-16) and (2-18), we may see that

$$T\,\frac{\partial}{\partial T}\left(\frac{\tau_i}{T}\right) = \frac{\partial\sigma_i}{\partial T} + \frac{\partial\tau_i}{\partial T} \tag{2-19}$$

so that

$$\tau_i = -T\,\frac{\partial\sigma_i}{\partial T} \tag{2-20}$$

which, from the equations of state (2-9,10), is equivalent to saying that

$$T_i = \frac{1}{\rho}(3\lambda+2\mu)\alpha T \ , \qquad \text{for} \quad i = 1, 2 \text{ or } 3 \tag{2-21}$$

$$= 0 \qquad\qquad\qquad \text{for} \quad i = 4, 5 \text{ or } 6$$

We see that requirements (2-15) and (2-16) are satisfied by the τ_i thus determined. In addition, from equations (2-16) and (2-21),

$$\frac{\partial c}{\partial \epsilon_i} = T \frac{\partial}{\partial T}\left(\frac{\tau_i}{T}\right) = 0 \quad , \tag{2-22}$$

i.e., c is independent of any of the strains.

The differential heat added to a unit mass can now be written in terms of known quantities

$$\rho \, dq = (3\lambda + 2\mu)\alpha T \, d\epsilon + \rho c \, dT \ . \tag{2-23}$$

Hence we see that c is just the specific heat of the material observed under constant strain conditions.

By assuming Fourier's Law for heat conduction,

$$J_i = -k T_{,i} \tag{2-24}$$

the field equation for T

$$k T_{,ii} = \rho c \frac{\partial T}{\partial t} + (3\lambda+2\mu)\alpha T \frac{\partial \epsilon}{\partial t} \tag{2-25}$$

is finally obtained. Of course, k is the thermal conductivity of the material. The inclusion of the last term differentiates equation (2-25) from the ordinary heat conduction equation derived by neglecting the thermal expansion of the material. The non-linearity of this extra term plays an important role, as we shall see later, in the determination of thermoelastic dissipation in steady-state vibrations of elastic bodies. We shall only point out here that a non-linear field equation has been derived from linear thermodynamic considerations. When T in the last term of equation (2-25) is taken to be constant, the equation forms the basis of so-called "linear thermoelasticity."

The heat conduction equation (2-25), the equations of motion (2-2), the equations of state (2-1), the compatibility requirements, and the appropriate stress, displacement, and temperature boundary conditions form the basic boundary value problem of the area of study commonly classified under the heading of thermoelasticity.

The solution of this problem, in the linearized version, is suggested in reference (5), along with the now well-known variational principle of thermoelasticity which provides a powerful tool for the numerical evaluation of solutions.

We note that for a given problem the equations of motion can be solved, in principle at least, in terms of the temperatures. The dilatation ϵ can then be substituted into the heat conduction equation which will contain the temperature as the only unknown.

Physically, we see that a temperature variation gives rise to a dilatation which plays the role of a heat source, causing a change of the temperature and further deformations.

III. THERMOELASTIC DAMPING

Inasmuch as all thermoelastic processes involve the conduction of heat, they are dissipative in nature. We shall discuss this point in detail.

Let us consider the rate of change of specific entropy $\frac{\partial s}{\partial t}$. Equations (2-8) and (2-11) can be combined into the equation

$$\rho \frac{\partial s}{\partial t} = - \frac{1}{T} J_{i,i} \tag{3-1}$$

which may be written in the form of a so-called "balance equation:" (7)

$$\rho \frac{\partial s}{\partial t} = -\left(\frac{J_i}{T}\right)_{,i} - \frac{1}{T^2}(T_{,i} J_i) \tag{3-2}$$

The change of the specific entropy is seen to come from two causes: the negative divergence of an entropy flow $\frac{J_i}{T}$ and an entropy production whose strength is given by the last term of the balance equation. Denote these two parts by $\rho \frac{d s_e}{d t}$ and $\rho \frac{d s_i}{d t}$

respectively, then

$$\frac{\partial s}{\partial t} = \frac{\partial s_e}{\partial t} + \frac{\partial s_i}{\partial t} \tag{3-3}$$

The local entropy production is, in the sense of general irreversible thermodynamics, the product of a "driving force,"

$$X_i = -\frac{T_{,i}}{T^2} \tag{3-4}$$

and a flux, in this case the heat flux J_i. If a linear phenomenological law is assumed, i.e.,

$$\begin{aligned} J_i &= L X_i \\ &= -\frac{L}{T^2} T_{,i} \end{aligned} \tag{3-5}$$

we obtain Fourier's Law of heat conduction, usually written in the form

$$J_i = -k T_{,i} \quad . \tag{3-6}$$

The conductivity k is, therefore, identified with $-\frac{L}{T^2}$. Equation (3-6) has been used in the derivation of the temperature field equation (2-25).

The local, or internal entropy production is a measure of the irreversibility of the process. It is a positive definite quantity. We see from equation (3-2) that the production of entropy in the case of thermoelasticity originates from the presence of a temperature gradient. The cause, and the only explicit cause, of dissipation is, therefore, the flow of heat from a higher temperature toward a lower temperature.

In a stationary state (7), the entropy, being a function of state, remains constant. The first member of equation (3-3) vanishes. Since the same amount of heat is entering the system at a higher temperature as is leaving at a lower temperature, more entropy is flowing out of the system and this amount must be made up locally in order to fulfill equation (3-3).

We shall direct our attention to a specific aspect of the general analysis of irreversible processes involving elastic solids. This is the area which may be loosely described by the term thermoelastic damping. To distinguish it from the study of damping in real materials, we remark that thermoelastic damping does not include the study of the many known causes or mechanisms of "conversion" of work into heat flow, for which bridged microscopic and macroscopic theories are needed. Rather, it comprises a study of what happens during heat flow due to mechanical deformations in an idealized, perfect continuum. Looking at it from a different viewpoint, this study is directed toward an answer to the question: How much of the total damping capacity is due to microscopic imperfections in a given, real material?

To be more specific, let the volume occupied by an elastic body, and bounded by the surface S, be V. Let this body be initially in thermodynamic equilibrium, i.e.,

$$\sigma_{ij} = \epsilon_{ij} = 0 \ ,$$

and $$T = T_o$$

at time $$t = 0 \quad .$$

For t > 0, periodic bodily forces and surface tractions are applied. If the boundary conditions for the temperature are time-independent, and if steady-state response can eventually be reached, we shall attempt to determine the net amount of mechanical energy required for each cycle to maintain this steady-state.

The system defined above is analogous to a "heat pump." The working medium, the elastic solid, is adjoined to an infinite reservoir at constant temperature T_o through conducting and insulating boundaries. When this system is being "operated" by the action of the applied forces, a temperature field different from T_o will be observed in V, inducing heat flow and increasing the total entropy of the system plus the reservoir.

Hence, any operation of this pump, excepting pure shear, will be an irreversible process. A net amount of positive work must be done on the system even in steady-state operations where the temperature and the strains return to their respective values after each cycle. From the first law, we may conclude that a net amount of heat must be "pumped" out of the system for each cycle. Therefore, we can immediately deduce that that the average temperature over a cycle must be raised above T_o . This is accomplished, of course, during the transient stages before the

steady-state is achieved.

The work done by the applied body forces and surface tractions per cycle is given by

$$
\begin{aligned}
W_o &= \int_t^{t+\frac{2\pi}{\omega}} \left[\int_V \rho F_i \dot{u}_i \, dv + \int_S \overset{\nu}{T}_i \dot{u}_i \, ds \right] dt \\
&= \int_t^{t+\frac{2\pi}{\omega}} \int_V \left[\mu \frac{\partial}{\partial t} (\epsilon_{ij} \epsilon_{ij}) + \frac{\lambda}{2} \frac{\partial}{\partial t} (\epsilon_{ii} \epsilon_{jj}) \right. \\
&\quad \left. - (3\lambda + 2\mu) \alpha \theta \frac{\partial \epsilon}{\partial t} \right] dv \, dt + \int_t^{t+\frac{2\pi}{\omega}} \int_V \frac{\partial}{\partial t} (\rho \dot{u}_i) \, dv \, dt
\end{aligned}
\tag{3-7}
$$

Also

$$
\overset{\nu}{T}_i = \sigma_{ij} \nu_j \qquad \text{on} \quad S \tag{3-8}
$$

In the above equations ω is the angular frequency of excitation and ν_j is the unit normal vector on S, pointed outward. By the divergence theorem,

$$
\begin{aligned}
\int_S \sigma_{ij} \dot{u}_i \nu_j \, ds &= \int_V (\sigma_{ij} \dot{u}_i)_{,j} \, dv \\
&= \int_V \sigma_{ij} \dot{u}_{i,j} \, dv + \int_V \sigma_{ij,j} \dot{u}_i \, dv
\end{aligned}
$$

and by the equations of motion the last term may be combined with the integral of the work done by the body forces,

$$
\int_V \rho F_i \dot{u}_i \, dv + \int_V \sigma_{ij,j} \dot{u}_i \, dv = \int_V \frac{\partial}{\partial t} (\rho \dot{u}_i) \, dv \tag{3-10}
$$

The equations of state have also been used in deriving equation (3-7).

Assuming the deformations are small, V and S may be considered independent of time, and the order of integration may be interchanged in equation (3-7),

$$W_o = \int_V \int_t^{t+\frac{2\pi}{\omega}} \left[\mu \frac{\partial}{\partial t}(\epsilon_{ij}\,\epsilon_{ij}) + \frac{\lambda}{2}\frac{\partial}{\partial t}(\epsilon_{ii}\,\epsilon_{jj}) \right] dt\; dv$$

$$-\int_V \int_t^{t+\frac{2\pi}{\omega}} (3\lambda+2\mu)\alpha\theta \frac{\partial\epsilon}{\partial t}\, dt\, dv + \int_V \int_t^{t+\frac{2\pi}{\omega}} \frac{\partial}{\partial t}(\rho\, \dot{u}_i)\, dt\; dv \; . \tag{3-11}$$

Because of periodicity, the first and the last integrals vanish and

$$W_o = -\int_t^{t+\frac{2\pi}{\omega}} \int_V (3\lambda+2\mu)\,\alpha\,\theta \frac{\partial\epsilon}{\partial t}\, dv\; dt \; . \tag{3-12}$$

The total heat flow out of V per cycle, H_o, can be evaluated by keeping account of the flow across S. It is given by the expression

$$H_o = -k\int_t^{t+\frac{2\pi}{\omega}} \int_s \theta_{,i}\; \nu_i\;\; ds\;\; dt \;\; . \tag{3-13}$$

Using the divergence theorem again, we obtain

$$H_o = -\int_t^{t+\frac{2\pi}{\omega}} \int_V k\;\, \theta_{,ii}\; dv\; dt \;\; . \tag{3-14}$$

From equations (2-25) and (3-14),

$$H_o = -\int_t^{t+\frac{2\pi}{\omega}} \int_V \left[(3\lambda+2\mu)\alpha(T_o+\theta)\frac{\partial\epsilon}{\partial t} + \rho\, c\, \frac{\partial\theta}{\partial t} \right] dv\, dt \tag{3-15}$$

or

$$H_o = -\int_t^{t+\frac{2\pi}{\omega}} \int_V (3\lambda + 2\mu)\, \alpha\, \theta\, \frac{\partial \epsilon}{\partial t}\, dv\, dt \quad . \tag{3-16}$$

Thus we have the identity

$$W_o = H_o \tag{3-17}$$

as expected.

An interesting point has been brought out in the above derivation. We note that if equation (2-25), the heat conduction relationship had been linearized, i.e., if

$$k\, T'_{,ii} = \rho\, c\, \frac{\partial T'}{\partial t} + (3\lambda + 2\mu)\, \alpha\, T_o\, \frac{\partial \epsilon'}{\partial t} \tag{3-18}$$

is used in equation (3-15), we obtain, instead of equation (3-16),

$$H'_o = 0 \quad . \tag{3-19}$$

The value of W_o is not affected by such linearization. We are led to an apparent contradiction to the first law,

$$W_o \neq H'_o \quad . \tag{3-20}$$

In equations (3-18), (3-19), and (3-20) the prime is used to associate the quantity in question to T', the temperature field defined by the linearized equation (3-18). Since the absolute temperature T is not expected to vary appreciably from T_o, (as the thermal boundary conditions are time independent), the non-linearity in equation (2-25) is indeed small, and T' should be a good approximation for T. But as just point out, one must exercise care if the energy dissipation per cycle of steady-state vibration is sought.

IV. AN EXAMPLE

The exact evaluation of thermoelastic energy dissipation requires solving simultaneously for the stress and temperature fields from the coupled equations of motion and heat conduction. We shall demonstrate some of the typical procedures involved in such an evaluation with the aid of a one-dimensional example of an infinite plate with uniform normal pressures applied sinusoidally on both flat surfaces. From the solution of this example some important principles are deduced, leading to the development of a general theory in Section V.

Let us choose the coordinate system such that the plane $X_1 = 0$ coincides with the mid-plane of the plate. If the plate thickness is 2h and the applied pressure is P sin ω t, the plate will be deforming in an extension-contraction mode perpendicular to the plate. The lowest natural frequency of the plate vibrating in this mode, determined by assuming isothermal conditions, is given by the well-known expression

$$\omega_o = \frac{\pi}{2h}\sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{4-1}$$

In the example, we will assume that the temperatures on both surfaces of the plate are to be kept at the constant reference temperature T_o for all time. Since all the quantities are functions of time and X_1 only, and since

$$u_2 = u_3 = 0 \quad , \tag{4-2}$$

the strains

$$\epsilon_{22} = \epsilon_{33} = \epsilon_{23} = \epsilon_{31} = \epsilon_{12} = 0 \quad , \tag{4-3}$$

and

$$\epsilon_{11} = u_{1,1} \quad . \tag{4-4}$$

The cubical dilatation is, therefore, simply

$$\epsilon = \epsilon_{11} \quad . \tag{4-5}$$

The stresses are

$$\sigma_{12} = \sigma_{23} = \sigma_{31} = 0 \; , \tag{4-6}$$

$$\sigma_{22} = \sigma_{33} = \lambda\epsilon - (3\lambda + 2\mu)\alpha\theta, \tag{4-7}$$

and
$$\sigma_{11} = (\lambda + 2\mu)\epsilon - (3\lambda + 2\mu)\alpha\theta \, . \tag{4-8}$$

The equations of motion(2-2) are reduced to only one,

$$\sigma_{11,1} = \rho \ddot{u}_1 , \tag{4-9}$$

in which the variation of the density is assumed small. By limiting ourselves to the case where the exciting frequency ω is well under ω_o, say $\omega \leqq 0.1\omega_o$, the right-hand side of equation (4-9) is extremely small, implying that σ_{11} is almost independent of X_1. Let us assume that

$$\sigma_{11,1} = 0$$

Since σ_{11} is equal to the applied loads on the boundary, it must be so everywhere within the plate thickness,

$$\sigma_{11} = P \sin \omega t \quad . \tag{4-10}$$

This means, from equation (4-8), that

$$\epsilon = \frac{P}{\lambda + 2\mu} \sin \omega t + \frac{3\lambda + 2\mu}{\lambda + 2\mu} \alpha \theta \tag{4-11}$$

The heat conduction equation (2-25) can now be reduced to

$$k \frac{\partial^2 \theta}{\partial x^2} = \rho c \frac{\partial \theta}{\partial t} + (3\lambda + 2\mu)\alpha(T_o + \theta)\left[\frac{P\omega \cos \omega t}{\lambda + 2\mu} + \frac{3\lambda + 2\mu}{\lambda + 2\mu} \alpha \frac{\partial \theta}{\partial t}\right] . \tag{4-12}$$

The solution θ must also satisfy the boundary conditions

$$\theta(\pm h, t) = 0 \tag{4-13}$$

for all t. Since we are seeking for the steady-state solution of θ, the condition of periodicity

$$\theta(X_1, t) = \theta(X_1, t + 2\pi/\omega) \tag{4-14}$$

must be applied for all values of X_1 within the plate.

We may non-dimensionalize equation (4-12) by introducing the dimensionless variables ξ, τ and ϕ which are defined as follows:

$$\xi = \frac{X_1}{h} \tag{4-15}$$

$$\tau = \omega t, \tag{4-16}$$

and

$$\phi = \frac{\theta}{T_o} \quad . \tag{4-17}$$

Upon substitution, we obtain the equation

$$\frac{\partial^2 \phi}{\partial \xi^2} = c_1 \frac{\partial \phi}{\partial t} + b_1 (1+\phi) \cos\tau + a_1 (1+\phi) \frac{\partial \phi}{\partial \tau} \quad . \tag{4-18}$$

The coefficients c_1, a_1, and b_1, are also non-dimensional and

$$c_1 = \frac{\rho c h^2 \omega}{k} \tag{4-19}$$

$$b_1 = \frac{P(3\lambda + 2\mu) a h^2 \omega}{(\lambda + 2\mu) k} \quad , \tag{4-20}$$

and

$$a_1 = \frac{(3\lambda + 2\mu)^2 a^2 T_o h^2 \omega}{(\lambda + 2\mu) k} \quad . \tag{4-21}$$

The boundary condition (4-13) in terms of the new quantities is

$$\phi(\pm 1, \tau) = 0 \quad , \tag{4-22}$$

and the periodicity condition (4-14) is now

$$\phi(\xi, \tau) = \phi(\xi, \tau + 2\pi) \tag{4-23}$$

for all ξ and τ .

For an idealized elastic material having physical properties similar to the common carbon steels, the ratios among c_1 , a_1 , and b_1 assume the proportionalities

$$c_1 : a_1 : b_1 = 400 : 1 : 1 \tag{4-24}$$

when the pressure amplitude, P, is as high as 14,000 psi, actually equivalent to more than a half of the stress magnitude needed to cause the material to yield. Notice that equation (4-24) holds independently of the quantities k, h, and ω . Since the induced temperature change, θ , is not expected to be large, it is convenient to make the transformation

$$\phi = \frac{b_1}{c_1} \psi \tag{4-25}$$

in equation (4-18) for the purpose of comparing relative magnitudes of the terms on the right-hand side of the equation. The magnitude of ψ is now of the order unity. Then

$$\frac{\partial^2 \psi}{\partial \xi^2} = c_1 \frac{\partial \psi}{\partial \tau} + c_1 (1 + \beta \psi) \cos \tau + a_1 (1 + \beta \psi) \frac{\partial \psi}{\partial \tau} \tag{4-26}$$

where $\beta = \dfrac{b_1}{c_1}$ (4-27)

has been used. The boundary and periodicity conditions are now, respectively,

$$\psi(\pm 1, \tau) = 0 \, , \tag{4-28}$$

and $\psi(\xi,\tau) = \psi(\xi,\tau+2\pi)$. (4-29)

Since

$$|\psi| \approx 1$$

all non-linear terms in equation (4-26) may be considered small.

The non-linear boundary value problem defined by equations (4-26), (4-28), and (4-29) will be solved by a perturbation scheme. The method will depend on the smallness of the two quantities β and $\frac{a_1}{c_1}$ (as compared to unity). Let us expand ψ into a power series of the parameter β ;

$$\psi(\xi,\tau) = \sum_{n=0}^{\infty} \psi_n(\xi,\tau)\beta^n , \tag{4-30}$$

and then substitute into both sides of the differential equation (4-26) to obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial\xi^2}(\psi_0 + \beta\psi_1 + \beta^2\psi_2 + \cdots\cdots) \\
&= c_1\frac{\partial}{\partial\tau}(\psi_0 + \beta\psi_1 + \beta^2\psi_2 + \cdots\cdots) + c_1\cos\tau \\
&\quad + a_1\frac{\partial}{\partial\tau}(\psi_0 + \beta\psi_1 + \beta^2\psi_2 + \cdots\cdots) \\
&\quad + c_1\beta(\psi_0 + \beta\psi_1 + \beta^2\psi_2 + \cdots\cdots)\cos\tau \\
&\quad + a_1\beta(\psi_0 + \beta\psi_1 + \beta^2\psi_2 + \cdots\cdots)\frac{\partial}{\partial\tau}(\psi_0 + \psi_1\beta + \psi_2\beta^2 + \cdots\cdots) .
\end{aligned} \tag{4-31}$$

Terms containing like powers of β on both sides of the last equation may be collected and equated, giving a set of linear differential equations for the coefficients of the power series.

$$\frac{\partial^2\psi_0}{\partial\xi^2} = (c_1 + a_1)\frac{\partial\psi_0}{\partial\tau} + c_1\cos\tau \tag{4-32}$$

$$\frac{\partial^2 \psi_1}{\partial \xi^2} = (c_1 + a_1)\frac{\partial \psi_1}{\partial \tau} + c_1 \psi_0 \cos\tau + a_1 \psi_0 \frac{\partial \psi_0}{\partial \tau} \quad , \tag{4-33}$$

$$\frac{\partial^2 \psi_2}{\partial \xi^2} = (c_1 + a_1)\frac{\partial \psi_2}{\partial \tau} + c_1 \psi_1 \cos\tau + a_1\left[\psi_0 \frac{\partial \psi_1}{\partial \tau} + \psi_1 \frac{\partial \psi_0}{\partial \tau}\right] ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \tag{4-34}$$

The problem is therefore reduced to solving the above set of differential equations in the given order. The boundary and periodicity conditions are, respectively,

$$\psi_n(\pm 1, \tau) = 0 \tag{4-35}$$

and

$$\psi_n(\xi, \tau) = \psi_n(\xi, \tau + 2\pi) \tag{4-36}$$

for each of the coefficients.

For n = 0, the solution ψ_0 may be found by writing

$$\psi_0(\xi, \tau) = f_0(\xi)\cos\tau + \left[g_0(\xi) - \frac{c_1}{c_1'}\right]\sin\tau \quad , \tag{4-37}$$

where

$$c_1' = c_1 + a_1 \quad . \tag{4-38}$$

and substituting into equation (4-32). Equating terms which contain the common factor $\cos\tau$, we get

$$f_0''(\xi) - c_1' g_0(\xi) = 0 \quad ; \tag{4-39}$$

equating terms which contain the common factor $\sin\tau$ we get

$$g_0''(\xi) + c_1' f_0(\xi) = 0 \quad . \tag{4-40}$$

The boundary conditions are

$$f_o(\pm 1) = 0 \quad , \tag{4-41}$$

$$g_o(\pm 1) = \frac{c_1}{c_1'} \tag{4-42}$$

The solutions for f_o and g_o are

$$f_o(\xi) = A_o \sin\sqrt{\frac{c_1'}{2}}\,\xi \sinh\sqrt{\frac{c_1'}{2}}\,\xi - B_o \cos\sqrt{\frac{c_1'}{2}}\,\xi \cosh\sqrt{\frac{c_1'}{2}}\,\xi \ , \tag{4-43}$$

and

$$g_o(\xi) = A_o \cos\sqrt{\frac{c_1'}{2}}\,\xi \cosh\sqrt{\frac{c_1'}{2}}\,\xi + B_o \sin\sqrt{\frac{c_1'}{2}}\,\xi \sinh\sqrt{\frac{c_1'}{2}}\,\xi \ , \tag{4-44}$$

with

$$A_o = \frac{c_1 \cos\sqrt{\frac{c_1'}{2}} \cosh\sqrt{\frac{c_1'}{2}}}{c_1'\left(\sinh^2\sqrt{\frac{c_1'}{2}} + \cos^2\sqrt{\frac{c_1'}{2}}\right)} \tag{4-45}$$

and

$$B_o = \frac{c_1 \sin\sqrt{\frac{c_1'}{2}} \sinh\sqrt{\frac{c_1'}{2}}}{c'\left(\sinh^2\sqrt{\frac{c_1'}{2}} + \cos^2\sqrt{\frac{c_1'}{2}}\right)} \tag{4-46}$$

which may be verified by direct substitution.

In the case of low carbon steel, we find for the non-dimensional coefficient c_1 the value of 4.5×10^5, taking h = 1 inch and $\omega = 30\% \, \omega_o$. Based on this knowledge, we can make the

following estimate of the magnitude of f_o (ξ)

$$|f_o(\xi)| \leq |A_o| \sinh\sqrt{\frac{c_1'}{2}}\,\xi + |B_o| \cosh\sqrt{\frac{c_1'}{2}}\,\xi$$

$$\leq \frac{\cosh\sqrt{\frac{c_1'}{2}}\,\sinh\sqrt{\frac{c_1'}{2}} + \sinh\sqrt{\frac{c_1'}{2}}\,\cosh\sqrt{\frac{c_1'}{2}}}{\sinh^2\sqrt{\frac{c_1'}{2}}} \approx 2\ ; \tag{4-47}$$

and, similarly, for g_o (ξ)

(4-48)

also

$$\psi_o^2 \leq f_o^2(\xi) + \left[g_o(\xi) - \frac{c_1}{c_1'}\right]^2$$

$$\leq f_o^2(\xi) + g_o^2(\xi) + 2\frac{c_1}{c_1'}\,|g_o(\xi)| + \left|\frac{c_1}{c_1'}\right|^2 = 4+4+4+1 = 13, \tag{4-49}$$

or $\quad |\psi_o| \leq 3.36$ approximately.

The magnitude of

$$\phi = \frac{\theta}{T_o}$$

is, therefore, limited to

$$|3.36 \times \beta| \leq 0.0084\ ,$$

or $\quad \theta \leq 0.0084 \times 273° = 2.2°F$.

Since the fraction $\frac{a_1}{c_1}$ is about $\frac{1}{400}$, and with the above limitations on the magnitude of ψ_o , the differential equation for ψ_1 may be simplified by omitting terms which are proportional to a_1 when the same functions appear with c_1 as a factor. Remembering that the solution for ψ_1 will be multiplied by the

factor β in the final solution ψ, it is reasonable to assume that this approximation will not introduce serious errors in the total solution. With this simplification, equation (4-33) becomes

$$\frac{\partial^2 \psi_1}{\partial \xi^2} = c_1' \frac{\partial \psi_1}{\partial \tau} + c_1 \psi_0 \cos \tau \; . \tag{4-51}$$

The solution is

$$\psi_1 = -\frac{c_1}{c_1'} \psi_0 \sin \tau + f_1(\xi) \cos 2\tau + \left[g_1(\xi) - \frac{c_1^2}{4c_1'^2} \right] \sin 2\tau \tag{4-52}$$

where

$$f_1(\xi) = A_1 \sin\sqrt{c_1'}\,\xi \, \sinh\sqrt{c_1'}\,\xi - B_1 \cos\sqrt{c_1'}\,\xi \, \cosh\sqrt{c_1'}\,\xi \quad ; \tag{4-53}$$

$$g_1(\xi) = A_1 \cos\sqrt{c_1'}\,\xi \, \cosh\sqrt{c_1'}\,\xi + B_1 \sin\sqrt{c_1'}\,\xi \, \sinh\sqrt{c_1'}\,\xi \quad , \tag{4-54}$$

with

$$A_1 = \frac{c_1^2 \cos\sqrt{c_1'} \, \cosh\sqrt{c_1'}}{4c_1'^2 (\sinh^2\sqrt{c_1'} + \cos^2\sqrt{c_1'})} \quad ; \tag{4-55}$$

$$B_1 = \frac{c_1^2 \sin\sqrt{c_1'} \, \sinh\sqrt{c_1'}}{4c_1'^2 (\sinh^2\sqrt{c_1'} + \cos^2\sqrt{c_1'})} \quad . \tag{4-56}$$

Additional perturbations may be obtained in like manner. They are not necessary, however, due to the rapid convergence of the series of powers of β. The magnitudes of f_1, g_1, and ψ_1 can be estimated and are comparable to f_0 and g_0. We may summarize by stating that the solution is very accurately represented by taking only the leading term of the series (4-30), and that there appears to be no need to go beyond a two-term approximation for the solution.

The work done during a cycle is given by the general expression derived in Section III:

$$W_o = -(3\lambda + 2\mu)\, a \int_t^{t+\frac{2\pi}{\omega}} \int_V \theta \frac{\partial \epsilon}{\partial t}\, dv\, dt \qquad (4\text{-}57)$$

Choosing the volume with a unit area perpendicular to the X_1 -axis as the volume V, and inserting expression (4-11) for ϵ into the above, we have

$$W_o = -\frac{(3\lambda+2\mu)\, a P \omega}{\lambda + 2\mu} \int_t^{t+\frac{2\pi}{\omega}} \int_{-h}^{h} \theta\, dX_1\, dt$$

$$= -\frac{(3\lambda+2\mu)^2 a^2 T_o P^2 h}{(\lambda+2\mu)^2 \rho c} \int_\tau^{\tau+2\pi} \cos\tau \int_{-1}^{1} \psi(\xi,\tau)\, d\xi\, d\tau \qquad (4\text{-}58)$$

It is seen that only those terms which contain the factor $\cos\tau$ in the solution for ψ contribute toward damping, excluding all terms except $f_o(\xi)\cos\tau$. The energy dissipation is, therefore, given by

$$W_o = -\frac{(3\lambda+2\mu)^2 a^2 T_o P^2 h}{(\lambda+2\mu)\rho c} \int_\tau^{\tau+2\pi} \cos^2\tau \int_{-1}^{1} f_o(\xi)\, d\xi\, d\tau$$

$$= \frac{(3\lambda+2\mu)^2 a^2 T_o P^2 h \pi}{2(\lambda+2\mu)^2 \rho c} \frac{c_1}{c_1'} \frac{(\sinh\sqrt{2c_1'} - \sin\sqrt{2c_1'})}{\sqrt{\frac{c_1'}{2}}\left(\sinh^2\sqrt{\frac{c_1'}{2}} + \cos^2\sqrt{\frac{c_1'}{2}}\right)}$$

$$= \frac{(3\lambda+2\mu)^2 a^2 T_o P^2 h \pi}{(\lambda+2\mu)^2 \rho c \sqrt{\frac{c_1'}{2}}} \qquad (4\text{-}59)$$

As a check, formula (3-14) may be used to compute the heat flow across the two unit surface areas,

$$H_o = -k\int_t^{t+\frac{2\pi}{\omega}} \int_V \theta_{,ii}\; dv\; dt$$

$$= \frac{-k\,T_o\,\beta}{h\,\omega}\int_\tau^{\tau+2\pi}\int_{-1}^{1} \frac{\partial^2\psi}{\partial\xi^2}\, d\xi\; d\tau$$

By equation (4-30),

$$H_o = -\frac{k\,T_o\beta}{h\omega}\int_\tau^{\tau+2\pi}\int_{-1}^{1}\left(\frac{\partial^2\psi_o}{\partial\xi^2} + \beta\frac{\partial^2\psi_1}{\partial\xi^2} + \beta^2\frac{\partial^2\psi_2}{\partial\xi^2} + \cdots\cdots\right) d\xi\; d\tau$$

Using equations (4-32), (4-33), (4-34), etc.,

$$H_o = -\frac{k\,T_o\beta}{h\omega}\int_\tau^{\tau+2\pi}\int_{-1}^{1}\left[(c_1+a_1)\frac{\partial\psi_o}{\partial\tau} + c_1\cos\tau\right] d\xi\; d\tau$$

$$-\frac{k\,T_o\beta^2}{h\omega}\int_\tau^{\tau+2\pi}\int_{-1}^{1}\left[(c_1+a_1)\frac{\partial\psi_1}{\partial\tau} + c_1\psi_0\cos\tau + a_1\psi_0\frac{\partial\psi_o}{\partial\tau}\right] d\xi\, d\tau$$

$$-\frac{k\,T_o\beta^3}{h\omega}\int_\tau^{\tau+2\pi}\int_{-1}^{1}\left[(c_1+a_1)\frac{\partial\psi_2}{\partial\tau}\cdots\cdots + \cdots\cdots\right] d\xi\; d\tau \qquad (4\text{-}60)$$

Since the functions ψ_o , ψ_1 .... are all periodic, we see that H_o will be zero unless at least a two-term approximation for ψ is used, in which case, the net heat flow out would be exactly equal to the work done in a cycle. To arrive at a basis of comparison with the actual damping capacities of a real material in this mode of deformation, we express W_o as a percentage of the

maximum stored strain energy, E, which is reached during a cycle

$$E = \frac{1}{2}\int_V \sigma_{ij}\,\epsilon_{ij}\,dv = \frac{1}{2(\lambda+2\mu)}\int_{-h}^{h}\sigma_{11}^2\,dx_1$$

$$= \frac{P^2 h}{\lambda+2\mu} \quad , \tag{4-61}$$

hence

$$\frac{W_o}{E} \leq \frac{(3\lambda+2\mu)^2 a^2 T_o \pi}{(\lambda+2\mu)\rho c\sqrt{\frac{c_1'}{2}}} \tag{4-62}$$

which is approximately 0.02% for h = 1" and $\omega = 30\%\,\omega_o$. This is less than the damping which would occur in a real, imperfect material. An interesting problem would be to formulate a phenomenological law of internal damping in terms of the entropy production, which would probably lead to a heat conduction equation with additional source terms. The solution might not be any more difficult if the same kind of approximation which is used in our example is made. Care must, of course, be exercised so as to maintain the same stress-strain-temperature relationship.

Let us return to the expression for thermoelastic damping, equation (4-59). We note that W_o is dependent of the frequency of excitation only through the parameters c_1 and c_1' , and that

$$\lim_{\omega\to 0} W_o = \lim_{\omega\to\infty} W_o = 0 \tag{4-63}$$

W_o is a positive, continuous function of the exciting frequency. This means that there exists a frequency at which the energy dissipated per cycle is at a maximum. The factor

$$\frac{\sinh\sqrt{\frac{c_1'}{2}}\cosh\sqrt{\frac{c_1'}{2}} - \sin\sqrt{\frac{c_1'}{2}}\cos\sqrt{\frac{c_1'}{2}}}{\sqrt{\frac{c_1'}{2}}\left(\sinh^2\sqrt{\frac{c_1'}{2}} + \cos^2\sqrt{\frac{c_1'}{2}}\right)}$$

is shown in Figure 1 on page 31. We see that the maximum is reached at a very low frequency, $\omega/\omega_0 \approx 0.01$ for this example, and that it is reached within a very narrow frequency band.

V. GENERAL THEORY

Within the range of linear thermodynamics, the dilatation ϵ may be written as the sum of two parts: the fractional change of volume due to the temperature and the fractional change of volume due to the stresses, i.e.,

$$\epsilon = \epsilon'(\theta) + \epsilon'' . \tag{5-1}$$

It is known that $\epsilon'(\theta)$ is linearly related to θ :

$$\epsilon'(\theta) = \lambda' \theta + \mu' \overline{\theta} . \tag{5-2}$$

where $\overline{\theta}$ is the average temperature over the entire volume at any given instant; and λ' and μ' are constants containing the elastic moduli.

The general heat equation (2-25) may now be written in terms of ϵ' and ϵ'' :

$$k\,\theta_{,ii} = \rho c \frac{\partial \theta}{\partial t} + (3\lambda + 2\mu)\alpha T_0 \left(\frac{\partial \epsilon''}{\partial t} + \frac{\theta}{T_0}\frac{\partial \epsilon''}{\partial t} + \frac{\partial \epsilon'}{\partial t} + \frac{\theta}{T_0}\frac{\partial \epsilon'}{\partial t} \right) . \tag{5-3}$$

We shall regard all terms in the parentheses as heat sources. The solution θ is known to be small. Let us examine the equation:

$$k\nabla^2\theta = \rho c \frac{\partial \theta}{\partial t} + (3\lambda + 2\mu)\alpha T_0 \left[\frac{\partial \epsilon''}{\partial t} + \frac{\theta'}{T_0}\frac{\partial \epsilon''}{\partial t} + \frac{\partial \epsilon'(\theta')}{\partial t} + \frac{\theta'}{T_0}\frac{\partial \epsilon(\theta')}{\partial t} \right] \tag{5-4}$$

in which θ' denotes a known function of small magnitude when compared to T_0 . The steady-state solution, θ , of equation (5-4) will be mainly due to the first of the heat sources,

$$(3\lambda + 2\mu)\,\alpha\, T_0 \frac{\partial \epsilon''}{\partial t} .$$

By neglecting all other sources, we obtain an approximate heat equation

$$k\,\theta_{,ii} = \rho c \frac{\partial \theta}{\partial t} + (3\lambda + 2\mu)\,\alpha\, T_o \frac{\partial \epsilon''}{\partial t} \tag{5-5}$$

which is linear, and which is uncoupled from the equations of motion.

The energy dissipation per cycle is a steady-state condition is given by the double integral [equation (3-12)] ,

$$W_o = (3\lambda + 2\mu)\,\alpha \int_t^{t+2\pi/\omega} \int_V \theta \frac{\partial \epsilon}{\partial t}\, dv\; dt \tag{5-6}$$

$$= (3\lambda + 2\mu)\,\alpha \int_t^{t+2\pi/\omega} \int_V \theta \frac{\partial}{\partial t}(\epsilon' + \epsilon'')\, dv\; dt$$

since

$$\int_t^{t+2\pi/\omega} \theta \frac{\partial \epsilon'}{\partial t}\, dt = 0$$ from periodicity considerations ,

$$W_o = -(3\lambda + 2\mu)\,\alpha \int_t^{t+2\pi/\omega} \int_V \theta \frac{\partial \epsilon''}{\partial t}\, dv\; dt \tag{5-7}$$

Multiply both sides of equation (3-5) by θ / T_o and integrate over V and $2\pi / \omega$. We see that the last integral becomes

$$(3\lambda + 2\mu)\,\alpha \int_t^{t+2\pi/\omega} \int_V \theta \frac{\partial \epsilon''}{\partial t}\, dv\, dt = \frac{k}{T_o} \int_t^{t+2\pi/\omega} \int_V \theta\, \theta_{,ii}\, dv\, dt\, . \tag{5-8}$$

By the use of the divergence theorem,

$$\int_V \theta\, \theta_{,ii}\, dv = \int_s \theta\, \theta_{,i}\, \nu_i\, ds - \int_V (\theta_{,i})^2 dv\ . \tag{5-9}$$

Equation (5-8) is transformed into an approximate formula for the quantity of work done per cycle of steady-state deformation:

$$W_o = \frac{k}{T_o}\int_t^{t+2\pi/\omega}\int_V (\theta_{,i})^2 dv\, dt \,. \tag{5-10}$$

Since either θ or $\theta_{,i}$ vanishes on the boundary S by the stationary boundary condition assumption.

By means of equation (5-10), the computation of thermoelastic damping in the general case is reduced to a) finding the dilatation by assuming isothermal conditions, b) determining the temperature field from equation (5-5), and c) evaluating the work done by formula (5-10). This is a considerable simplification over the "exact method" which has been illustrated by the example in Section IV. In that example, it may be shown easily that the above approximate formula will lead to a result which is the same as the "exact" formula except that c_1' is replaced by c_1 in equation (4-59). It may, therefore, be said that equation (5-10) furnishes a much simplified route to calculating the amount of energy dissipation for small deformations.

VI. CONCLUSIONS

The following conclusions are reduced from observing the results and their derivations in the above discussion:

(1) The field equations of thermoelasticity are derived by considering the conservation of mass, momentum, and energy In addition, a second law postulated for perfect elastic materials is found necessary. These equations represent a phenomenological theory consistent with the notions of continuum mechanics in the study of elastic deformations and heat flow.

(2) Thermoelastic damping in idealized materials is the explicit result of the irreversible character of heat conduction only.

(3) For real materials, it is suggested that the numerous macroscopically observed damping laws may be represented by source terms in the heat conduction equation.

(4) The heat conduction equation for perfect materials is non-linear although a linear thermodynamic process has been considered.

(5) An inconsistency will appear between the solutions of the field equations and the first law if the heat equation is linearized in the manner of the usual thermoelasticity theories in the literature.

(6) In processes where thermal boundary conditions are stationary in time, the general expression for the energy dissipation for a steady-state cycle of deformation is derived.

(7) A simple one-dimensional example is given in Section IV. The equations of motion (equilibrium) are solved in terms of the temperature field, which is in turn solved from the non-linear heat conduction equation by a perturbation scheme. For an idealized elastic material having properties similar to low carbon steel, the said non-linearity is shown to be unnecessary only if the energy dissipation is computed by the work done on the system.

(8) In the given example, the induced temperature variation is small (less than 2.2°F for a maximum stress of 14,000 psi, and at a frequency which is 30 percent of the first natural frequency of the system). The amount of energy dissipated during a cycle is also extremely small (about 0.02 percent of the maximum strain energy reached during a cycle). Based on these facts, the heat equation is linearized and uncoupled by neglecting all source terms except that representing the rate of change of dilatation, calculated by assuming isothermal deformations.

(9) The above simplification is generalized to derive an expression for the energy dissipation in arbitrary bodies under elastic deformations.

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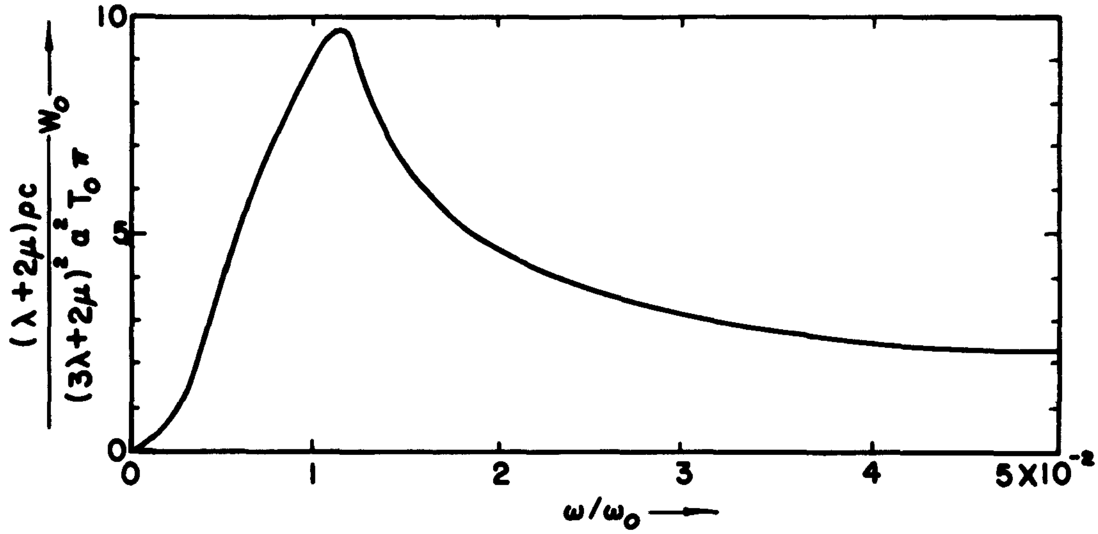

Figure 1. - Thermoelastic Damping vs Exciting Frequency

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Rpt Nr ASD-TDR-62-1031 THERMOELASTIC DAMPING. Summary report, Dec 62, 31 p. incl illus., 7 refs.

Unclassified report

The non-linear coupled field equations of thermoelasticity are herein solved for the quasi-static behavior of a solid bounded by two parallel planes. The mechanical energy converted to heat during each cycle of the loading process is explicitly evaluated and its spectral variation computed. From this exact solution an approximate, but quite

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